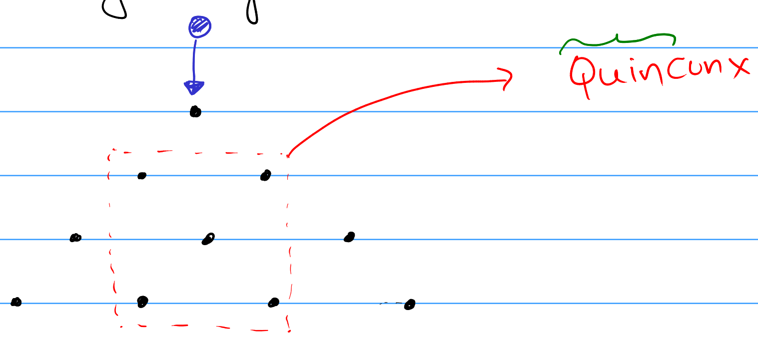
Introduction (Lecture 1)

Your introductory probability class should have ended with the Central Limit theorem.

Here is a physical demonstration: A Galton Board, invented in the 1800s by Francis Galton, Charles Darwin's cousin.

You can see many things in this:

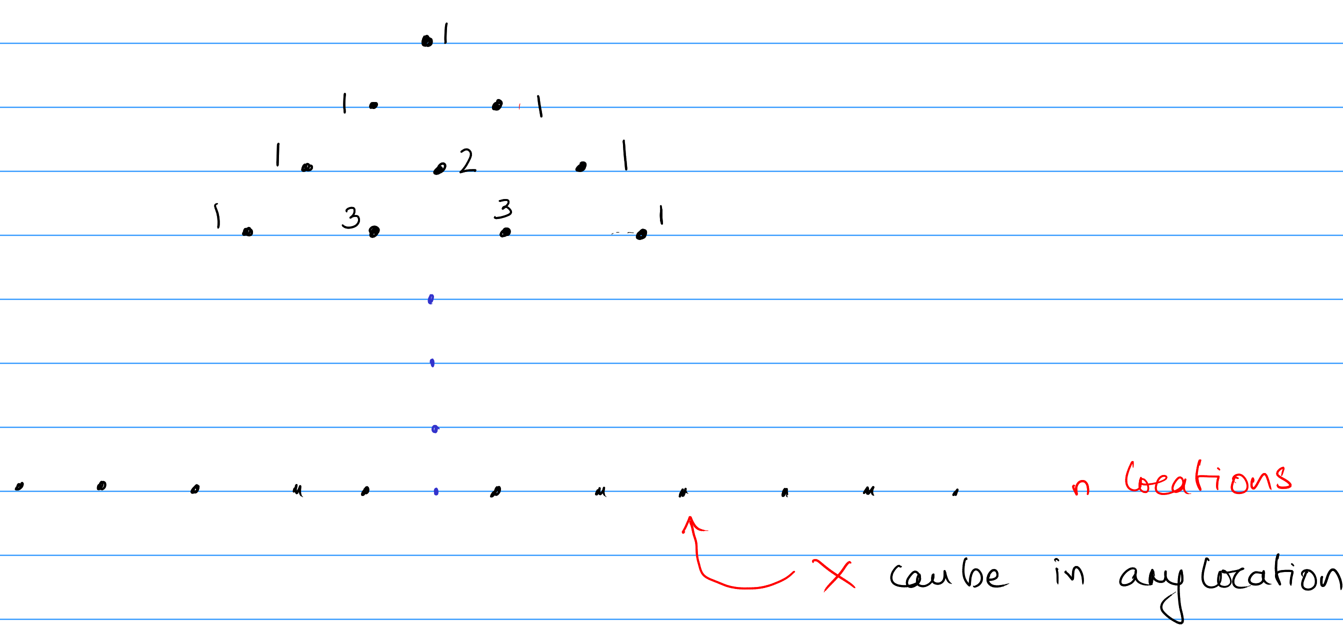


There are many things you can see

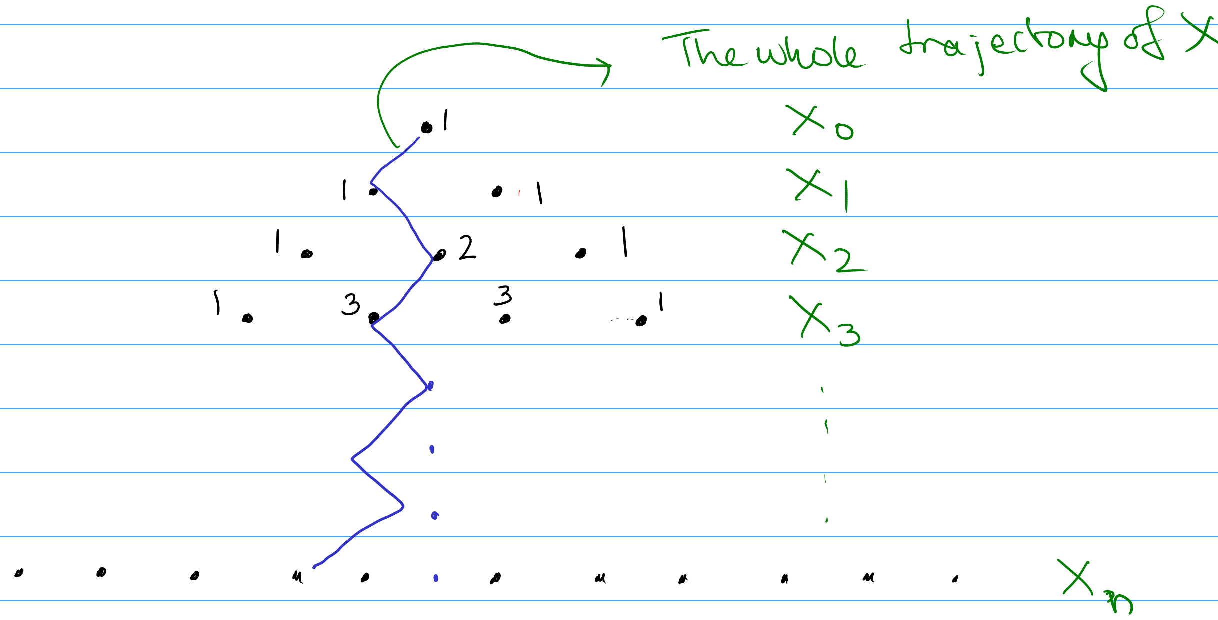
1) CLT (when many balls are run through it)

2) Pascal's triangle (Binomial distribution)

3) Random walk.



What is the distribution of X ?



$(X_0, X_1, \ldots, X_n)$ is a sequence of n random numbers that represent the trajectory of the ball!

This is a STOCHASTIC PROCESS:

$\{X_i\}_{i=1}^{n}$

i represents ____________

Review

Basic setup : Ω = {probability space}

$A \subset \Omega$ is an event

$\mathcal{F}$ = {set of all events}

Ex : $\Omega = \{HH, HT, TH, TT\}$ A = {1st coin toss is a head} = {HT, HH}

An experiment is the selection of an element from Ω.

What does this experiment represent?

$A \cap B$: If events A and B occur

$A \cup B$: If events A OR B occur

ϕ : empty set, impossible event

Ω : sure event

Prob meas or function

$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$

Ex : $\mathbb{P}(A) = \mathbb{P}(\text{1st coin toss is heads}) = \frac{2}{4}$

Countable additivity :

The events A and B are disjoint if $A \cap B = \phi$

or "if A and B cannot occur together"

$\{A_i\}_{i=1}^{\infty}$ are disjoint if they're pairwise disjoint, ie $A_i \cap A_j = \emptyset$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Law of total probability:

Let $A_1 = \{$1st coin toss is heads$\}$

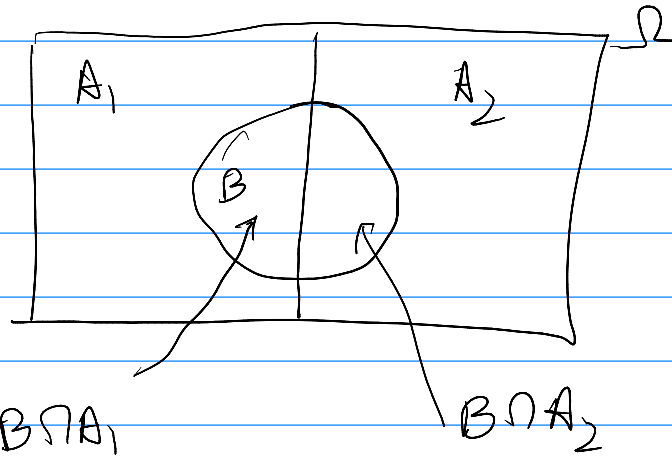
$A_2 = \{$1st " " tails$\}$

$B = \{$2nd coin toss is tails$\}$

Then

$$P(B) = \frac{1}{2} = P(B \cap A_1) + P(B \cap A_2)$$

In pictures



Independence: $\{A_i\}_{i=1}^{n}$ are indep if for any subcollection $\{A_{i_1}, \ldots, A_{i_k}\}$ we have

$$\prod_{j=1}^{k} P(A_{i_j}) = P(A_{i_1} \cap \cdots \cap A_{i_k})$$

Random Variables

What is a random variable? $X : \Omega \longrightarrow \mathbb{R}$

Discrete rvs: $X \in \{a_1, \ldots, \ldots\}$ countable

PMF $\quad P(X = a_i) = p(a_i) \quad i = 1, \ldots, \infty$

pmf

CDF $= P(X \leq t) = F(t) \quad F(t)$ jumps at a_i

$$p(a_i) = F(a_i) - F(a_i^-)$$

right limit   left limit

Continuous RVs

$$P(X = a) = 0 \quad \forall \; a \in \mathbb{R}$$

$\Rightarrow$ $F(a)$ is a continuous function of a.

Sometimes $\exists$ a function $f(s)$ st

$$F(a) = \int_{-\infty}^{a} f(s)\, ds$$

where $f(s)$ is called the pdf

If F is diff at a then

$$\left. \frac{dF(x)}{dx} \right|_{x=a} = f(a)$$

Examples: (1.3.1)

1) Bernoulli
2) Geometric
3) Binomial
4) Poisson

Bernoulli's occur as indicators of events. Let A be an event

$$\mathbb{1}_A := \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$P(\mathbb{1}_A = 1) = P(A) = p$$

$$\mathbb{1}_A \sim \text{Bernoulli}(p)$$

Binomials are sums of independent Bernoullis:

$$X \sim \text{Binomial}(n, p)$$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Geometrics represent the time of the first success as we keep tossing a coin.

$$Z \sim \text{Geometric}(p)$$

$$P(Z = k) = (1-p)^{k-1} p$$

Poisson: $X \in \{1, 2, \dots\}$

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Ex: (In class)

A fraction $p = 0.05$ of the items coming off a production line are defective. In a random sample of 10, what is the probability that the sample contains exactly 1 defective item?

Important Continuous Distributions (1.4.1)

1) Normal distribution

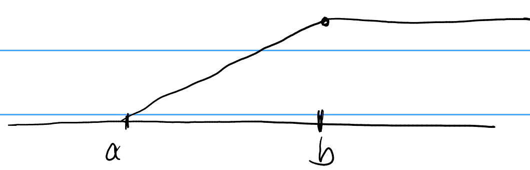
2) Exponential distributions

3) Uniform distributions

Review: pdf, cdf.

Uniform distribution

PDF $$f = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$



CDF

Mean $\frac{a+b}{2}$ Variance (not worth remembering!)

Moments

Discrete: $\mathbb{E}[X] = \sum_{i \geq 0}^{\infty} a_i \, p(a_i)$

k^{th} moment: $\mathbb{E}[X^k] = \sum_{i=0}^{\infty} a_i^k \, p(a_i)$

Variance: $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \geq 0$

written as σ^2 usually.

Continuous: $\mathbb{E}[X^m] = \int_{-\infty}^{\infty} x^m f(x)\, dx$

Suppose X is an rv. Then $Y = \sin(X)$ is also "random", so you can ask: what is its average value?

$$\mathbb{E}[\sin(X)] = \sum_{i=0}^{\infty} \underbrace{\sin(a_i)}_{\text{value}} \, \underbrace{p(a_i)}_{\text{probability}}$$

If continuous

$$\mathbb{E}[\sin(X)] = \int_{-\infty}^{\infty} \sin(x) f(x)\, dx$$

In general, if $Y = g(X)$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

Joint PDFs (1.2.4)

Given a pair of rvs (X, Y)

$$F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

if $\exists\, f_{XY}(\xi, \eta)$ st

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(\xi,\eta)\, d\xi\, d\eta \quad \Bigg| \quad \frac{\partial^2 F}{\partial x \partial y} = f_{X,Y}(x,y)$$

then its called joint pdf

Marginal of X (pdf of X)

$$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\, dv$$

Independence: If $F(x,y) = \overbrace{F(x)}^{\text{product.}}F(y)\ \forall x,y$

then X and Y are said to be independent.

Other forms:

$$P(X \in (a,b), Y \in (c,d)) = P(X \in (a,b))\, P(Y \in (c,d))$$

$$\forall a,b,c,d$$

If densities exist

$$f_{XY}(a,b) = f_X(a)\, f_Y(b)$$

Covariance and Correlation:

$$\mathrm{Cov}(X,Y) = E[(X - EX)(Y - EY)]$$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

$$= E[XY] - (EX)(EY)$$

If X, Y independent
$E[f(x)g(y)] = E[g(x)]E[h(y)]$

Independence $\Rightarrow$ $Cov(X,Y) = 0$

Independence $\not\Leftarrow$ $Cov(X,Y) = 0$
(unless variables are jointly gaussian.)

Moment generating functions

$$E[e^{tX}] = \sum_{i=0}^{\infty} p(a_i) e^{ta_i} = M(t) \text{ (discrete)}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = M(t)$$

$$M'(0) = E[X]$$
$$M^{(n)}(0) = E[X^n]$$

n^{th} moment given by n^{th} derivative of MGF at 0.

MGFs uniquely identify distributions.

MGFs behave well under convolution! (we will see this next)

Convolutions : (1.2.5)

If X, Y are independant and $Z = X+Y$:

Let $X \in \{a_1, a_2 \ldots\}$ $Y \in \{b_1, b_2, \ldots\}$

$$p_Z(u) = \mathbb{P}(Z=u) = \sum_{i=1}^{\infty} \mathbb{P}(Y = u - a_i, X = a_i)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(Y = u - a_i)\, \mathbb{P}(X = a_i)$$

$$= \sum_{i=1}^{\infty} p_X(u-a_i)\, p_Y(a_i)$$

2) Continuous X and Y have pdfs f_X and f_Y and are independ $Z = X+Y$

$$\Rightarrow f_Z(t) = \int_{-\infty}^{\infty} f_X(t-u)\, f_Y(u)\, du$$

$t - u + u = t$

3) MGFs

$$E[e^{tZ}] = M_Z(t) = E[e^{tX} e^{tY}]$$

$$= E[e^{tX}]\, E[e^{tY}]$$

$$= M_X(t)\, M_Y(t)$$

Ex. 1) Let $X \sim \text{Geom}(p)$ Find $M_X(t)$

$$M_X(t) = \sum_{k=1}^{\infty} e^{kt}(1-p)^{k-1} p = \sum_{k=1}^{\infty} \left((1-p)e^t\right)^k \frac{p}{1-p}$$

$$= \frac{p}{1-p}\left[1 - \frac{1}{1-(1-p)e^t}\right]$$

2) Let X, Y be independent geometrics with parameter p.

Find

$M_{X+Y}(t)$

What kind of random variable is $X+Y$?

$$X+Y \sim \text{Negbin}(\underset{\text{\# of successes}}{2}, p)$$

Conditional Probability (1.2.7)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

Better statement:

$$P(A \cap B) = P(A|B)\,P(B) = P(B|A)\,P(A)$$

given

Ex

Urn	Gold	Silver
I	3	4
II	2	7

Flip a coin to choose urns then pick a random ball.

$A = \{\text{Urn I}\}$

$B = \{\text{Gold coin}\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

$$P(B) = P(B|A)\,P(A) + P(B|A^c)\,P(A^c)$$

$$= \frac{3}{7}\,\frac{1}{2} + \frac{2}{9} \cdot \frac{1}{2}$$

Change of Variable (1.2.6)

Let $U \sim$ Uniform $[0,1]$ $\quad f_U(t) = \begin{cases} 1 & t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

Let $X = U^2$. Find the pdf of X.

In general, let $Y = g(X)$, and assume g is strictly increasing, and differentiable.

$$P(Y \le t) = F_Y(t) = P(g(X) \le t) = P(X \le g^{-1}(t))$$

$$= \int_{-\infty}^{g^{-1}(t)} f_X(u)\,du$$

$$\frac{d}{dt} F_Y(t) = f_X(g^{-1}(t)) \frac{d}{dt} g^{-1}(t)$$

How to compute $g^{-1}(t)$? $\quad g(g^{-1}(t)) = t$

By chain rule $\quad g'(g^{-1}(t)) \frac{d}{dt} g^{-1}(t) = 1$

$$\Rightarrow \quad \frac{d}{dt} g^{-1}(t) = \frac{1}{g'(g^{-1}(t))}$$

$$\Rightarrow \quad f_Y(t) = f_X(g^{-1}(t)) \frac{1}{g'(g^{-1}(t))}$$

Joint Normal Distribution (1.4.6)

Let X_i have mean and variance μ_i, σ_i^2 for $i = 1, 2$. Let $-1 < \rho < 1$ be the correlation coefficient of X_1, X_2.

We say (X_1, X_2) are jointly normal if their joint pdf is given by

$$\phi_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp \underbrace{\left[-\sum_{i=1}^{2} \frac{(x_i - \mu_i)^2}{2\sigma_i} + \frac{2\rho (x_1 - \mu_1)(x_2 - \mu_2)}{\sigma_1 \sigma_2} \right]}_{Q(x_1, x_2)}$$

Ex: What is the marginal distribution

$f_{X_1}(t)$

Ex. For what values of ρ are X_1 and X_2 independent?